

MODEL IN R^3 THE THREEDIMENSIONAL SPHERE AND CALCULATE ITS VOLUME

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Abstract. In the present note, on the basis of a result of G. Perelman, we offer a model in R^3 the three-dimensional sphere and calculate its volume using spherical coordinates. A simple computing scheme makes it easy to generalize it to n -dimensional case.

МОДЕЛЬ В R^3 ТРЕХМЕРНОЙ СФЕРЫ И ВЫЧИСЛЕНИЕ ЕЁ ОБЪЕМА

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Ключевые слова: гипотеза Пуанкаре, сфера Пуанкаре, сферические координаты.

Аннотация. В настоящей заметке, на основании результата Г. Перельмана, предлагается модель в R^3 трехмерной сферы и вычисление ее объема с помощью сферических координат. Простая схема вычисления позволяет легко обобщить ее для n -мерного случая.

The development of astronautics keeps up with the knowledge of outer space. The study of spaces of greater dimension is a necessary step in the study and knowledge of space. One of the first steps is associated with four-dimensional space [1-2] and its structure. Our goal is to popularize the “real” 3-sphere model and a fairly simple way to calculate the volume of a three-dimensional sphere. In 1904, Henri Poincaré suggested that any three-dimensional object that has certain properties of a three-dimensional sphere (3-sphere) can be transformed into a 3-sphere. (More precisely, any simply connected three-dimensional compact manifold without an edge is homeomorphic to a three-dimensional sphere). It took 99 years to prove this hypothesis. Russian mathematician Grigory Perelman proved the Poincare hypothesis put forward a hundred years ago and completed the creation of a catalog of forms of three-dimensional spaces.

In fact, the proof of Perelman, whose correctness has not yet been called into question by anyone, solves a much wider range of issues than the Poincare conjecture itself. The geometrization procedure proposed by William P. Thurston [3] from Cornell University allows a complete classification of three-dimensional manifolds, which is based on a three-dimensional sphere, unique in its sublime simplicity. If the Poincare conjecture were false, i.e. if there were many spaces as simple as a sphere, the classification of 3-manifolds would turn into something infinitely more complex. Thanks to Perelman and Thurston, we have a complete catalog of all the mathematically acceptable forms of three-dimensional space that our Universe could take (if we consider only space without time).

Let us return to the goal of our article and move on to constructing a “tangible” 3-sphere model. Let’s try to “see” how a 3-sphere is constructed from a 2-sphere and a 3-ball, after considering simpler cases.

A line is a 1-ball and its border is two points - a 0-sphere. We “bend” a segment into a semicircle and connecting the points of its boundary with the same “bent” segment, we obtain a circle. It limits a two-dimensional circle. T.O. we took two copies of the 1-ball, with the border in the form of two 0-spheres, deformed and glued them together on the 0-spheres, we got a 2-ball with the border of the 1-sphere. Those. The 1-sphere is a circle, and the 2-ball is a circle.

Repeat our design for. Take two instances of a 2-ball (flat circle), with a 1-sphere border (circle) and deform each 2-ball into a hemisphere, then glue these hemispheres along the border (1-sphere) - as a result, we get a 3-ball, with border 2 sphere.

Now it's easy for us to repeat our actions for. We act similarly, take 2 copies of a 3-ball, with a border consisting of a 2-sphere (we do not deform, we, unfortunately, are people from) and glue them together along the border of 2-spheres. The resulting body will be a 3-sphere. We cannot imagine it, it is impossible, it lies in, but we have a real, "tangible" model that allows us to work with the 3-sphere. We indicate a simple and intuitive way of calculating its area and volume.

Let B be a ball of radius R centered at the origin in the space of variables x, y, z, w. Then its volume can be calculated as an integral of multiplicity 4 over region B.

$$V(B) = \int \int \int \int_B dv = \int \int \int \int_B dx dy dz dw.$$

This integral is more convenient to calculate in spherical coordinates (see also [4]) r, φ , ψ , θ in four-dimensional space, which are introduced as follows:

$$x = r \cos \varphi \cos \theta \cos \psi; y = r \sin \varphi \sin \theta \cos \psi; z = r \sin \theta \cos \psi; w = r \sin \psi,$$

where $0 \leq r < \infty$, $0 \leq \varphi \leq 2\pi$, $-0.5\pi \leq \theta, \psi \leq 0.5\pi$.

It is clear how spherical coordinates are also introduced for an arbitrary n. But for the Jacobian of transition I, and in the case n = 4, painstaking calculations of the Jacobian are required. The calculation of this Jacobian can be done in a simpler way using the following system of equalities

$$\begin{cases} F_1 = r^2 - (x^2 + y^2 + z^2 + w^2) = 0, \\ F_2 = r^2 \cos^2 \psi - (x^2 + y^2 + z^2) = 0, \\ F_3 = r^2 \cos^2 \psi \cos^2 \theta - (x^2 + y^2) = 0, \\ F_4 = r^2 \cos^2 \varphi \cos^2 \psi \cos^2 \theta - x^2 = 0. \end{cases}$$

Obviously, it is equivalent to the introduced spherical (gyroscopic) coordinate system. We use the implicit function theorem in the extended sense (implicit mapping theorem). For the Jacobians, the equality

$$\frac{D(x, y, z, w)}{D(r, \varphi, \psi, \theta)} = (-1)^4 \frac{\frac{D(F_1, F_1, F_1, F_1)}{D(r, \varphi, \psi, \theta)}}{\frac{D(F_1, F_1, F_1, F_1)}{D(x, y, z, w)}}.$$

But the Jacobians on the right side have triangular matrices and are easily calculated

$$\left| \frac{D(F_1, F_1, F_1, F_1)}{D(x, y, z, w)} \right| = 16r^4 \cos \varphi \sin \varphi \cos^3 \psi \cdot \sin \psi \cdot \cos^2 \theta \cdot \sin \theta,$$

$$\left| \frac{D(F_1, F_1, F_1, F_1)}{D(r, \varphi, \psi, \theta)} \right| = 16r^7 \cos \varphi \sin \varphi \cos^5 \psi \sin \psi \cos^3 \theta \sin \theta.$$

Therefore, $|I| = r^3 \cos^2 \psi \cos \theta$

$$\begin{aligned} V_4(B) &= \int \int \int \int_{Bxyzw} dv = \int \int \int \int_{B r \varphi \psi \theta} r^3 \cos^2 \psi \cos \theta dr d\varphi d\psi d\theta = \\ &= \int_0^{2\pi} d\varphi \int_0^R r^3 dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \psi d\psi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta = 2\pi \frac{R^4}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 + \cos 2\psi)}{2} d\psi \cdot (\sin \theta) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \\ &= 2\pi \frac{R^4}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 + \cos 2\psi)}{2} d\psi \cdot 2 = \pi \frac{R^4}{2} (\psi + \sin 2\psi) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi^2 R^4}{2}. \end{aligned}$$

Therefore, $V_4(B) = \frac{\pi^2 R^4}{2}$ and unit ball volume

$$V_4(B) = \frac{\pi^2 R^4}{2} \Big|_{R=1} = \frac{\pi^2}{2}.$$

Since the area of the sphere is a derivative of the volume of the ball along the radius, then

$$\frac{\partial V_4(B)}{\partial R} = \frac{\partial}{\partial R} \frac{\pi^2 R^4}{2} = 2\pi^2 R^3 = S_3(\partial B).$$

Thus $S_3(\partial B) = 2\pi^2 R^3$.

The calculation scheme is easily adaptable for arbitrary n. A different, original way is proposed in [5].

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